Crystal Structures on Bitableaux and the Kronecker Problem

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joint work with Nate Harman (UGA)

Outline

Section 1

Schur Modules and the Kronecker problem

Partitions and Tableaux

A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n is a list of weakly decreasing positive integers summing to n. For example $(5, 3, 3, 1)$ is a partition of 12.

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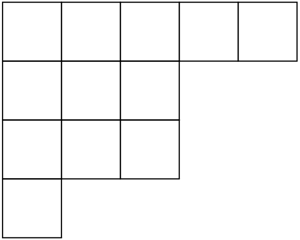

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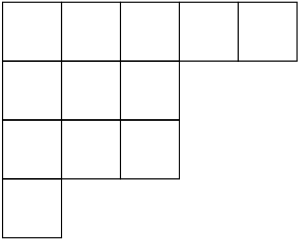

A tableau of shape λ is a filling of λ's Young diagram with objects.

Schur Modules

Given a vector space V with basis $\{v_1, \ldots, v_n\}$, write $S^\lambda(V)$ for the vector space with basis indexed by tableaux of shape λ whose boxes are filled with basis elements v_i whose indices weakly increase along rows and strictly increase down columns.

Example

Two of the following tableaux correspond to basis vectors

v_1	v_1	v_2
v_2	v_3	

v_1	v_4	v_5
v_3	v_4	

v_1	v_2	v_2
v_2	v_3	

v_1	v_2	v_1
v_2	v_3	

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Example

Two of the following tableaux correspond to basis vectors

1	1	2
2	3	

1	4	5
3	4	

1	2	2
2	3	

1	2	1
2	3	

These are called semistandard Young tableaux.

Schur Modules

The Schur module $S^\lambda(V)$ is a representation of $GL(V)$. This means that there is an action of $GL(V)$ (which we can think of as $n \times n$ matrices) on $S^\lambda(V)$.

Schur Modules

Let $V = \mathbb{R}^2$. Then a 2×2 matrix acts on $S^\lambda(V)$ in the following way.

$$
\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \cdot
\begin{array}{|c|c|}\hline v_1 & v_1 \\ \hline v_2 \\ \cline{1-1}\end{array}
=
\begin{array}{|c|c|}\hline 2v_2 & 2v_2 \\ \hline v_1 \\ \cline{1-1}\end{array}
+
\begin{array}{|c|c|}\hline 2v_2 & 2v_2 \\ \hline 3v_2 \\ \cline{1-1}\end{array}
$$

$$
= 4\begin{array}{|c|c|}\hline v_2 & v_2 \\ \hline v_1 \\ \cline{1-1}\end{array}
+ 12\begin{array}{|c|c|}\hline v_2 & v_2 \\ \hline v_2 \\ \cline{1-1}\end{array}
$$

$$
= -4\begin{array}{|c|c|}\hline v_1 & v_2 \\ \hline v_2 \\ \cline{1-1}\end{array}
$$

Schur Modules

The modules $S^\lambda(V)$ for $\ell(\lambda) \leq \dim(V)$ are the irreducible (polynomial) representations of the group $GL(V)$. This means that any (polynomial) representation of $GL(V)$ decomposes into a direct sum of Schur modules.

The Kronecker Problem

Now let's consider two vector spaces V and W and their tensor product $V \otimes W$. The Schur module

$$S^\lambda(V \otimes W)$$

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This module must then have a decomposition into irreducible representations of the form $S^{\mu}(V) \otimes S^{\nu}(W)$:

$$S^{\lambda}(V \otimes W) \cong \bigoplus_{\mu,\nu} g_{\lambda,\mu,\nu} S^{\mu}(V) \otimes S^{\nu}(W).$$

The Kronecker Problem

The coefficients $g_{\lambda,\mu,\nu}$ in

$$S^{\lambda}(V \otimes W) \cong \bigoplus_{\mu,\nu} g_{\lambda,\mu,\nu} S^{\mu}(V) \otimes S^{\nu}(W)$$

are called the Kronecker coefficients. The Kronecker problem asks for a combinatorial interpretation for these coefficients.

Section 2

Lexicographic Bitableaux

Lexicographic Bitableaux

If V has basis $\{v_1, \ldots, v_n\}$ and W has basis $\{w_1, \ldots, w_m\}$, then $V \otimes W$ has basis

$$\{e_{(i,j)} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

where $e_{(i,j)} = v_i \otimes w_j$. We order this basis lexicographically where $(i_1, j_1) < (i_2, j2)$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$.

Lexicographic Bitableaux

Now, a basis for $S^\lambda(V \otimes W)$ consists of tableaux filled with pairs of positive integers which are semistandard with respect to the lexicographic order. We call these objects lexicographic bitableaux.

$e_{(1,2)}$	$e_{(1,3)}$	$e_{(1,3)}$	$e_{(2,1)}$
$e_{(2,3)}$	$e_{(2,2)}$		
$e_{(3,1)}$			

Lexicographic Bitableaux

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1 2	1 3	1 3	2 1
2 3	2 2		
3 1			

Section 3

Crystals

Crystals

Given a word w (i.e. a sequence of numbers), let $f_i(w)$ be defined as follows:

	12213213
(1) Replace each i and $i+1$ in w with) and (respectively.	1))1()1(
(2) Note the rightmost unmatched).	1))1()1(
(3) Swap the corresponding i with $i+1$.	12313213

We then write either

$$f_2(12213213) = 12313213$$

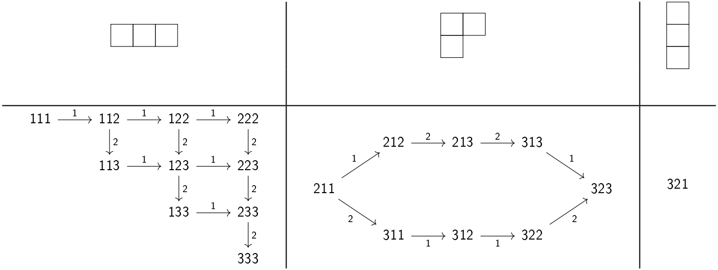
or

$$12213213 \xrightarrow{2} 12313213.$$

Connecting all words with a given max entry via these arrows gives a graph structure on words called a crystal.

Crystals

Each connected crystal on words corresponds to a unique tableau shape.



This crystal encodes the structure of the corresponding irreducible representation $S^\lambda(V)$.

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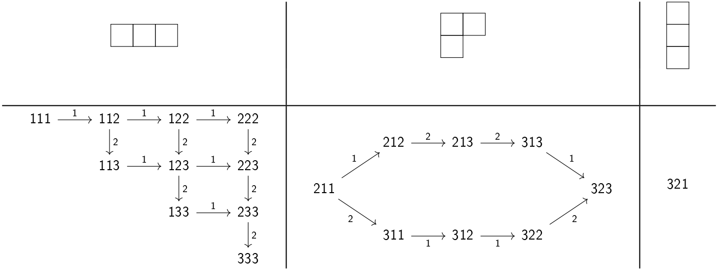
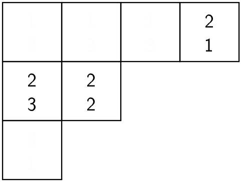


This crystal encodes the structure of the corresponding irreducible representation $S^{\lambda}(V)$.

Core idea: If you can put a crystal structure on the representation you want to study, the connected components tell you what the irreducible pieces are.

Crystals on Bitableaux

Given a bitableau T, we extract words $w^i(T)$ for $i \geq 1$. For example, to extract $w^2(T)$, we first ignore all boxes except for those with first entry 2.



Then we read the second entry from each box from left-to-right starting at the bottom row and moving to the top:

$$w^2(T) = 321$$

Crystals on Bitableaux

Putting all these words together, we extract a single word

$$w^{\bullet}(T) = w^1(T)w^2(T)\dots$$

$$T = \begin{array}{|c|c|c|c|} \hline \begin{smallmatrix}1\\2\end{smallmatrix} & \begin{smallmatrix}1\\3\end{smallmatrix} & \begin{smallmatrix}1\\3\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} \\ \hline \begin{smallmatrix}2\\3\end{smallmatrix} & \begin{smallmatrix}2\\2\end{smallmatrix} & \multicolumn{2}{c}{} \\ \cline{1-2} \begin{smallmatrix}3\\1\end{smallmatrix} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$w^{\bullet}(T) = 2333211.$$

Crystals on Bitableaux

To apply f_i to T, we simply apply the usual crystal operation to this word:

$$T = \begin{array}{|c|c|c|c|} \hline \begin{matrix}1\\2\end{matrix} & \begin{matrix}1\\3\end{matrix} & \begin{matrix}1\\3\end{matrix} & \begin{matrix}2\\1\end{matrix} \\ \hline \begin{matrix}2\\3\end{matrix} & \begin{matrix}2\\2\end{matrix} \\ \cline{1-2} \begin{matrix}3\\1\end{matrix} \\ \cline{1-1} \end{array}$$

$$w^\bullet(T) = 2333211$$

$$f_1(2333211) = 233321\textcolor{purple}{2}$$

$$f_1(T) = \begin{array}{|c|c|c|c|} \hline \begin{matrix}1\\2\end{matrix} & \begin{matrix}1\\3\end{matrix} & \begin{matrix}1\\3\end{matrix} & \begin{matrix}2\\1\end{matrix} \\ \hline \begin{matrix}2\\3\end{matrix} & \begin{matrix}2\\2\end{matrix} \\ \cline{1-2} \begin{matrix}3\\\textcolor{purple}{2}\end{matrix} \\ \cline{1-1} \end{array}$$

Takeaway

This method allows us to decompose $S^\lambda(V \otimes W)$ as *either* a $GL(V)$-representation *or* a $GL(W)$-representation.

To resolve the Kronecker problem, we need two crystal structures—one acting on the first entries, the other acting on second entries—that are compatible with each other.

THANK

YOU

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